

Vector F -implicit complementarity problems with corresponding variational inequality problems

B.S. Lee^{a,*}, M. Firdosh Khan^b, Salahuddin^b

^a *Department of Mathematics, Kyungshung University, Busan 608-736, Republic of Korea*

^b *Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*

Received 1 May 2006; accepted 9 May 2006

Abstract

In this work, we present some new existence theorems for solutions for a new class of generalized vector F -implicit complementarity problems and the corresponding generalized vector F -implicit variational inequality problems by using the Fan–KKM Theorem under some suitable assumptions without monotonicity, and establish the equivalence between those problems in Banach spaces.

© 2006 Elsevier Ltd. All rights reserved.

Keywords: Vector F -implicit complementarity problems; Vector F -implicit variational inequality problems; KKM-mapping; Fan–KKM theorem; Pointed cone

1. Introduction

Complementarity theory introduced by Lemke [17] and Cottle and Danzig [4] has been studied and applied in nonlinear analysis for more than forty years, from the 1960s [1–3,5,7–16,18,19]. In particular, there have been many discussions on the relations between the solution sets to complementarity problems and the solution sets to the corresponding variational inequality problems [1–5,7–12,14–16,18,19]. In [2], the authors considered the relation between the multivalued implicit complementarity problems and the multivalued implicit variational inequality problems. Cottle and Yao [5] considered some existences of solutions for a nonlinear complementarity problem involving a pseudo-monotonicity mapping on a closed convex cone in Hilbert spaces, and showed some necessary and sufficient conditions for the existence of solutions to some variational inequality problems. In 2001, Yin et al. [19] introduced a class of F -complementarity problems (F -CP) for finding $x \in K$ such that

$$\langle Tx, x \rangle + F(x) = 0 \quad \text{and} \quad \langle Tx, y \rangle + F(y) \geq 0 \quad \text{for all } y \in K$$

where K is a nonempty closed convex cone of a real Banach space X , $T : K \rightarrow X^*$, the dual space, is a mapping and $F : K \rightarrow (-\infty, +\infty)$ is a function, and proved that it is equivalent to the following generalized variational inequality problems:

* Corresponding author.

E-mail addresses: bslee@ks.ac.kr (B.S. Lee), khan_mfk@yahoo.com (M.F. Khan), salahuddin12@mailcity.com (Salahuddin).

Find $x \in K$ such that

$$\langle Tx, y - x \rangle + F(y) - F(x) \geq 0 \quad \text{for all } y \in K,$$

where F is a positively homogeneous and convex function. They also proved the existence of solutions for $(F\text{-CP})$ under some assumptions with F -pseudo-monotonicity.

In 2003, Fang and Huang [7] introduced and studied a new class of vector F -complementarity problems with demi-pseudomonotone mappings in Banach spaces by considering the solvability of the problems. Huang and Li [8] introduced and studied a class of scalar F -implicit complementarity problems and another class of F -implicit variational inequality problems in Banach spaces, in 2004. Last year, the result for a scalar case in [8] was extended and generalized to the vector case by Li and Huang [18]. The equivalence between the F -implicit complementarity problem and F -implicit variational inequality problem was presented and some new existence theorems for solutions for F -implicit variational inequality problems were also proved.

The main objective of this work is to generalize some results of [8,18] to a more generalized vector case. We introduce a new class of generalized vector F -implicit complementarity problems and a corresponding new class of generalized vector F -implicit variational inequality problems in Banach spaces and prove the equivalence between them under certain assumptions. Furthermore, we derive some new existence theorems of solutions for the generalized vector F -implicit complementarity problems and the generalized vector F -implicit variational inequality problems by using the Fan–KKM Theorem [6] under some suitable assumptions without any monotonicity.

2. Preliminaries

Let Y be a real Banach space. Let $P \subset Y$ be a nonempty closed convex and pointed cone with the apex at the origin, that is, P is a closed set with the following conditions;

- (i) $\lambda P \subseteq P$ for all $\lambda > 0$;
- (ii) $P + P \subseteq P$;
- (iii) $P \cap \{-P\} = \{0\}$.

An ordered Banach space (Y, P) is a real Banach space Y with an ordering defined by a closed cone $P \subseteq Y$ with an apex at the origin in the form of

$$x \geq y \Leftrightarrow x - y \in P$$

and

$$x \not\geq y \Leftrightarrow x - y \notin P.$$

We recall some definitions and lemmas needed in our work.

Definition 2.1. Let X and Y be vector spaces and K a subspace of X . A mapping $F : K \rightarrow Y$ is said to be positively homogeneous if $F(\alpha x) = \alpha F(x)$ for all $x \in K$ and $\alpha \geq 0$.

Definition 2.2. Let K be a nonempty subset of topological vector space X . A mapping $G : K \rightarrow 2^X$ is called a KKM-mapping if for every finite subset $\{x_1, x_2, \dots, x_n\}$ of K

$$\text{conv}\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^n G(x_i),$$

where conv denotes the convex hull.

Fan–KKM Theorem 2.1 ([6]). Let K be a nonempty subset of Hausdorff topological vector space X . Let $G : K \rightarrow 2^X$ be a KKM-mapping such that for any $y \in K$ $G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$; then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$, i.e.,

$$\bigcap_{y \in K} G(y) \neq \emptyset.$$

Lemma 2.2 ([18]). Let (Y, P) be an ordered Banach space induced by a pointed closed convex cone P . Then $x \geq 0$ and $y \geq 0$ imply that $x + y \geq 0$, for all $x, y \in Y$.

3. Main results

Throughout this section, let X be a real Banach space, $K \subseteq X$ be a nonempty closed convex cone and (Y, P) be an ordered Banach space induced by a pointed closed convex cone P . Denote as $L(X, Y)$ the space of all continuous linear mappings from X into Y and as $\langle t, x \rangle$ the value of a linear continuous mapping $t \in L(X, Y)$ at x . Let $A, T : K \rightarrow L(X, Y)$, $g : K \rightarrow K$, $F : K \rightarrow Y$ and $N : L(X, Y) \times L(X, Y) \rightarrow L(X, Y)$ be mappings. In this section, we consider the following generalized vector F -implicit complementarity problem (GVF-ICP): Find $x^* \in K$ such that

$$\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) = 0$$

and

$$\langle N(Ax^*, Tx^*), g(y) \rangle + F(g(y)) \geq 0, \quad \text{for all } y \in K. \quad (3.1)$$

Special cases

(1) The following vector F -implicit complementarity problem (VF-ICP) of finding $x^* \in K$ such that

$$\langle f(x^*), g(x^*) \rangle + F(g(x^*)) = 0$$

and

$$\langle f(x^*), y \rangle + F(y) \geq 0, \quad \text{for all } y \in K \quad (3.2)$$

is a particular form of (GVF-ICP) for the identities A and T and a mapping $f : K \rightarrow L(X, Y)$ defined by $f(x) = N(x, x)$ for $x \in K$, which was considered and studied by Li and Huang [18].

(2) If g is an identity mapping on K and $F \equiv 0$, then the (VF-ICP) collapses to the vector F -complementarity problem (VF-CP) of finding $x^* \in K$ such that

$$\langle f(x^*), x^* \rangle = 0 \quad \text{and} \quad \langle f(x^*), y \rangle \geq 0, \quad \text{for all } y \in K, \quad (3.3)$$

which was studied by Chen and Yang [3].

(3) If $L(X, Y) = X^*$ and $F : K \rightarrow \mathbb{R}$, then (VF-ICP) collapses to the scalar F -implicit complementarity problem (SF-ICP) of finding $x^* \in K$ such that

$$\langle f(x^*), g(x^*) \rangle + F(g(x^*)) = 0 \quad \text{and} \quad \langle f(x^*), y \rangle + F(y) \geq 0, \quad \text{for all } y \in K,$$

which was considered by Huang and Li [8].

Remark 3.1. Letting g be an identity (respectively, $F \equiv 0$) in (SF-ICP), we obtain complementarity problems considered in Yin et al. [19] (respectively, Isac [9,14]).

We also introduce the following generalized vector F -implicit variational inequality problem (GVF-IVIP): Find $x^* \in K$ such that

$$\langle N(Ax^*, Tx^*), g(y) - g(x^*) \rangle + F(g(y)) - F(g(x^*)) \geq 0, \quad \text{for all } y \in K. \quad (3.4)$$

The following vector F -implicit variational inequality problem (VF-IVIP) of finding $x^* \in K$ such that

$$\langle f(x^*), y - g(x^*) \rangle + F(y) - F(g(x^*)) \geq 0, \quad \text{for all } y \in K$$

is a particular form of (GVF-IVIP), which was considered in [18].

We first establish the equivalence between (GVF-ICP) and (GVF-IVIP).

Theorem 3.1. (i) If x^* solves (GVF-ICP), then x^* solves (GVF-IVIP).

(ii) If $F : K \rightarrow Y$ is a positively homogeneous mapping and x^* solves (GVF-IVIP), then x^* solves (GVF-ICP).

Proof. (i) Let $x^* \in K$ be a solution of (GVF-ICP); then

$$\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) = 0$$

and

$$\langle N(Ax^*, Tx^*), g(y) \rangle + F(g(y)) \geq 0 \quad \text{for all } y \in K.$$

It follows that

$$\begin{aligned} & \langle N(Ax^*, Tx^*), g(y) - g(x^*) \rangle + F(g(y)) - F(g(x^*)) \\ &= [\langle N(Ax^*, Tx^*), g(y) \rangle + F(g(y))] - [\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*))] \\ &= \langle N(Ax^*, Tx^*), g(y) \rangle + F(g(y)) \\ &\geq 0, \end{aligned}$$

for all $y \in K$; thus x^* is a solution of (GVF-IVIP).

(ii) Let $x^* \in K$ be a solution of (GVF-IVIP); then

$$\langle N(Ax^*, Tx^*), g(y) - g(x^*) \rangle + F(g(y)) - F(g(x^*)) \geq 0 \quad \text{for all } y \in K.$$

Since $F : K \rightarrow Y$ is a positively homogeneous mapping and K is a convex cone, letting $g(y) = 2g(x^*)$ and $g(y) = \frac{1}{2}g(x^*)$ in (3.4), we have

$$\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) \geq 0$$

and

$$\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) \leq 0,$$

that is,

$$\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) \in P \cap \{-P\}.$$

Since P is a pointed cone

$$\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) = 0.$$

Moreover, we obtain

$$\begin{aligned} \langle N(Ax^*, Tx^*), g(y) \rangle + F(g(y)) &= [\langle N(Ax^*, Tx^*), g(y) - g(x^*) \rangle + F(g(y)) - F(g(x^*))] \\ &\quad + [\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*))] \\ &= \langle N(Ax^*, Tx^*), g(y) - g(x^*) \rangle + F(g(y)) - F(g(x^*)) \\ &\geq 0 \end{aligned}$$

for all $y \in K$. Hence x^* solves (GVF-ICP). \square

If A, T and g are identity mappings on K , then we have the following result as a corollary.

Corollary 3.1 ([18]). (i) If x^* solves (VF-ICP), then x^* solves (VF-IVIP).

(ii) If $F : K \rightarrow Y$ is positively homogeneous and x^* solves (VF-IVIP), then x^* solves (VF-ICP).

Now we consider the existence of solutions to (GVF-IVIP) and the properties of the solution sets.

Theorem 3.2. Assume that

(a) five mappings $N : L(X, Y) \times L(X, Y) \rightarrow L(X, Y)$, $g : K \rightarrow K$, $A, T : K \rightarrow L(X, Y)$ and $F : K \rightarrow Y$ are continuous;

(b) there exists a mapping $h : K \times K \rightarrow Y$ such that

- (i) $h(x, x) \geq 0$ for all $x \in K$;
- (ii) $\langle N(Ax, Tx), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) - h(x, y) \geq 0$ for all $x, y \in K$;
- (iii) the set $\{y \in K : h(x, y) \not\geq 0\}$ is convex for all $x \in K$;

(c) there exists a nonempty compact convex subset C of K such that for all $x \in K \setminus C$ there exists $y \in C$ such that

$$\langle N(Ax, Tx), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \not\geq 0.$$

Then GVF-IVIP has a solution. Furthermore, the solution set of (GVF-IVIP) is closed.

Proof. Define a set-valued mapping $G : K \rightarrow 2^C$ by

$$G(y) = \{x \in C : \langle N(Ax, Tx), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \geq 0\}, \quad \text{for all } y \in K.$$

By the assumption (a), for any $y \in K$, $G(y)$ is closed in C . Since every element $x^* \in \bigcap_{y \in K} G(y)$ is a solution of (GVF-IVIP), we have to show that $\bigcap_{y \in K} G(y) \neq \emptyset$. Since C is compact it is sufficient to prove that the family $\{G(y)\}_{y \in K}$ has the finite intersection property. Let $\{y_1, y_2, \dots, y_n\}$ be a finite subset of K and set $B := \overline{\text{conv}}(C \cup \{y_1, y_2, \dots, y_n\})$. Then B is a compact and convex subset of K .

Define two set-valued mappings $F_1, F_2 : B \rightarrow 2^B$ as follows:

$$F_1(y) = \{x \in B : \langle N(Ax, Tx), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \geq 0\} \quad \text{for all } y \in B,$$

and

$$F_2(y) = \{x \in B : h(x, y) \geq 0\} \quad \text{for all } y \in B.$$

From the conditions (i) and (ii), we have

$$h(y, y) \geq 0$$

and

$$\langle N(Ay, Ty), g(y) - g(y) \rangle + F(g(y)) - F(g(y)) - h(y, y) \geq 0.$$

Now Lemma 2.2 implies

$$\langle N(Ay, Ty), g(y) - g(y) \rangle + F(g(y)) - F(g(y)) \geq 0$$

and so $F_1(y)$ is nonempty. Similarly, we can prove that for any $y \in K$, $F_1(y)$ is closed. Since $F_1(y)$ is a closed subset of a compact set B , we know that $F_1(y)$ is compact. Next, we prove that F_2 is a KKM-mapping. Suppose that there exists a finite subset $\{u_1, u_2, \dots, u_n\}$ of B and $\lambda_i \geq 0$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \lambda_i = 1$ such that

$$u = \sum_{i=1}^n \lambda_i u_i \notin \bigcup_{j=1}^n F_2(u_j).$$

Then

$$h(u, u_j) \not\geq 0, \quad j = 1, 2, \dots, n.$$

From the condition (iii), we have

$$h(u, u) \not\geq 0,$$

which contradicts the condition (i). Hence F_2 is a KKM-mapping. On the other hand, from the condition (ii), we have

$$F_2(y) \subseteq F_1(y) \quad \text{for all } y \in B.$$

In fact, $x \in F_2(y)$ implies that $h(x, y) \geq 0$ and by the condition (ii), we have

$$\langle N(Ax, Tx), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) - h(x, y) \geq 0.$$

It follows from Lemma 2.2 that

$$\langle N(Ax, Tx), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \geq 0,$$

i.e., $x \in F_1(y)$. Thus F_1 is also a KKM-mapping. From Fan–KKM Theorem 2.1, there exists $x^* \in B$, such that $x^* \in F_1(y)$ for all $y \in B$. That is, there exists $x^* \in B$ such that

$$\langle N(Ax^*, Tx^*), g(y) - g(x^*) \rangle + F(g(y)) - F(g(x^*)) \geq 0 \quad \text{for all } y \in B.$$

By assumption (c), we get $x^* \in C$ and moreover $x^* \in G(y_i)$, $i = 1, 2, \dots, n$. Hence $\{G(y)\}_{y \in K}$ has the finite intersection property.

Since $A, T : K \rightarrow X^*$, $g : K \rightarrow K$, $F : K \rightarrow Y$ and $N : X^* \times X^* \rightarrow X^*$ are continuous, the solution set of (GVF-IVIP) is obviously closed. \square

Remark 3.2. Letting g be an identity in Theorem 3.2, we obtain the same result for the following vector F -implicit variational inequality problem: Find $x \in K$ satisfying

$$\langle N(Ax^*, Tx^*), y - x^* \rangle + F(y) - F(x^*) \geq 0 \quad \text{for all } y \in K.$$

Theorem 3.3. Assume that $A, T : K \rightarrow L(X, Y)$, $N : L(X, Y) \times L(X, Y) \rightarrow L(X, Y)$ and $g : K \rightarrow K$ are continuous and $F : K \rightarrow Y$ is positively homogeneous and continuous. If assumptions (b) and (c) in Theorem 3.2 hold, then (GVF-ICP) has a solution. Furthermore, the solution set of (GVF-ICP) is closed.

Proof. The conclusion follows directly from Theorems 3.1 and 3.2. \square

Corollary 3.2 ([18]). Assume that

- (a) $f : K \rightarrow L(X, Y)$, $g : K \rightarrow K$ and $F : K \rightarrow Y$ are continuous;
- (b) there exists a mapping $h : K \times K \rightarrow Y$ such that

- (i) $h(x, x) \geq 0$, for all $x \in K$;
- (ii) $\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) - h(x, y) \geq 0$ for all $x, y \in K$;
- (iii) the set $\{y \in K : h(x, y) \not\geq 0\}$ is convex, for all $x \in K$;

- (c) there exists a nonempty, compact, convex subset C of K such that for all $x \in K \setminus C$, there exists $y \in C$ such that

$$\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) \not\geq 0.$$

Then (VF-IVIP) has a solution. Furthermore, the solution set is closed.

References

- [1] A. Carbone, A note on complementarity problem, *Internat. J. Math. Math. Sci.* 21 (3) (1998) 621–623.
- [2] S.S. Chang, N.J. Huang, Generalized multivalued implicit complementarity problems in Hilbert spaces, *Math. Japonica* 36 (6) (1991) 1093–1100.
- [3] G.Y. Chen, X.Q. Yang, The vector complementarity problems and its equivalences with the weak minimal element in ordered spaces, *J. Math. Anal. Appl.* 153 (1990) 136–158.
- [4] R.W. Cottle, G.B. Danzig, Complementarity pivot theory of mathematical programming, *Linear Algebra Appl.* 1 (1968) 103–125.
- [5] R.W. Cottle, J.C. Yao, Pseudo-monotone complementarity problems in Hilbert space, *J. Optim. Theory Appl.* 75 (2) (1992) 281–295.
- [6] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* 142 (1961) 305–310.
- [7] Y.P. Fang, N.J. Huang, The vector F -complementarity problem with demipseudomonotone mappings in Banach spaces, *Appl. Math. Lett.* 16 (2003) 1019–1024.
- [8] N.J. Huang, J. Li, F -implicit complementarity problems in Banach spaces, *Z. Anal. Anwendungen* 23 (2004) 293–302.
- [9] G. Isac, A special variational inequality and the implicit complementarity problems, *J. Fac. Sci. Univ. Tokyo* 37 (1990) 109–127.
- [10] G. Isac, *Complementarity Problems*, Springer-Verlag, New York, 1992.
- [11] G. Isac, A generalization of Karamardian's condition in complementarity theory, *Nonlinear Anal. Forum* 4 (1999) 49–63.
- [12] G. Isac, On the implicit complementarity problem in Hilbert spaces, *Bull. Austral. Math. Soc.* 32 (1985) 251–260.
- [13] G. Isac, Condition $(S)_+^1$, Altman's condition and the scalar asymptotic derivative: Applications to complementarity theory, *Nonlinear Anal. Forum* 5 (2000) 1–13.
- [14] G. Isac, *Topological Methods in Complementarity Theory*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
- [15] G. Isac, J. Li, Complementarity problems, Karamardian's condition and a generalization of Harker–Pang condition, *Nonlinear Anal. Forum* 6 (2) (2001) 383–390.
- [16] S. Karamardian, Generalized complementarity problem, *J. Optim. Theory Appl.* 8 (1971) 161–168.
- [17] C.E. Lemke, Bimatrix equilibrium points and mathematical programming, *Manage. Sci.* 11 (1965) 681–689.
- [18] J. Li, N.J. Huang, Vector F -implicit complementarity problems in Banach spaces, *Appl. Math. Lett.* 19 (2006) 464–471.
- [19] H.Y. Yin, C.X. Xu, Z.X. Zhang, The complementarity problems and its equivalence with the least element problem, *Acta Math. Sinica* 44 (2001) 679–686.